

ACD Term Rewriting

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Abstract. In this paper we introduce Associative Commutative Distributive Term Rewriting (ACDTR), a rewriting language for rewriting logical formulae. ACDTR extends AC term rewriting by adding *distribution* of conjunction over other operators. Conjunction is vital for expressive term rewriting systems since it allows us to require that multiple conditions hold for a term rewriting rule to be used. ACDTR uses the notion of a “conjunctive context”, which is the conjunction of constraints that must hold in the context of a term, to enable the programmer to write very expressive and targeted rewriting rules. ACDTR can be seen as a general logic programming language that extends Constraint Handling Rules and AC term rewriting. In this paper we define the semantics of ACDTR and describe our prototype implementation.

1 Introduction

Term rewriting is a powerful instrument to specify computational processes. It is the basis of functional languages; it is used to define the semantics of languages and it is applied in automated theorem proving, to name only a few application areas.

One difficulty faced by users of term rewriting systems is that term rewrite rules are *local*, that is, the term to be rewritten occurs in a single place. This means in order to write precise rewrite rules we need to gather all relevant information in a single place.

Example 1. Imagine we wish to “program” an overloaded ordering relation for integers variables, real variables and pair variables. In order to write this the “type” of the variable must be encoded in the term¹ as in:

$$\begin{aligned} int(x) \leq int(y) &\rightarrow intleq(int(x), int(y)) \\ real(x) \leq real(y) &\rightarrow realleq(real(x), real(y)) \\ pair(x_1, x_2) \leq pair(y_1, y_2) &\rightarrow x_1 \leq y_1 \vee x_1 = y_1 \wedge x_2 \leq y_2 \end{aligned}$$

In a more standard language, the type information for variables (and other information) would be kept separate and “looked up” when required. \square

¹ Operator precedences used throughout this paper are: \wedge binds tighter than \vee , and all other operators, e.g. \neg , $=$, bind tighter than \wedge .

Term rewriting systems such as constraint handling rules (CHRs) [5] and associative commutative (AC) term rewriting [3] allow “look up” to be managed straightforwardly for a single conjunction.

Example 2. In AC term rewriting the above example could be expressed as:

$$\begin{aligned} int(x) \wedge int(y) \wedge x \leq y &\rightarrow int(x) \wedge int(y) \wedge inteq(x, y) \\ real(x) \wedge real(y) \wedge x \leq y &\rightarrow real(x) \wedge real(y) \wedge realleq(x, y) \\ pair(x, x_1, x_2) \wedge pair(y, y_1, y_2) \wedge x \leq y &\rightarrow pair(x, x_1, x_2) \wedge pair(y, y_1, y_2) \wedge \\ &\quad (x_1 \leq y_1 \vee x_1 = y_1 \wedge x_2 \leq y_2) \end{aligned}$$

where each rule replaces the $x \leq y$ by an appropriate specialised version, in the conjunction of constraints. The associativity and commutativity of \wedge is used to easily collect the required type information from a conjunction. \square

One difficulty remains with both AC term rewriting and CHRs. The “look up” is restricted to be over a single large conjunction.

Example 3. Given the term $int(x_1) \wedge int(y_1) \wedge pair(x, x_1, x_2) \wedge pair(y, y_1, y_2) \wedge x \leq y$. Then after rewriting $x \leq y$ to $(x_1 \leq y_1 \vee x_1 = y_1 \wedge x_2 \leq y_2)$ we could not rewrite $x_1 \leq y_1$ since the types for x_1, y_1 appear in a different level.

In order to push the type information inside the disjunction we need to distribute conjunction over disjunction. \square

Simply adding distribution rules like

$$A \wedge (B \vee C) \rightarrow A \wedge B \vee A \wedge C \quad (1)$$

$$A \wedge B \vee A \wedge C \rightarrow A \wedge (B \vee C) \quad (2)$$

does not solve the problem. Rule (1) creates two copies of term A , which increases the size of the term being rewritten. Adding Rule (2) to counter this effect results in a non-terminating rewriting system.

1.1 Conjunctive context

We address the non-termination vs. size explosion problem due to distributivity rewrite rules in a similar way to how commutativity is dealt with: by handling distributivity on the language level. We restrict ourselves to dealing with expanding distributivity of conjunction \wedge over any other operator, and we account for idempotence of conjunction.² Thus we are concerned with distribution rules of the form

$$P \wedge f(Q_1, \dots, Q_n) \rightarrow P \wedge f(P \wedge Q_1, \dots, P \wedge Q_n). \quad (3)$$

² This means that conjunction is distributive over any function f in presence of a redundant copy of P , i.e. $P \wedge (P \wedge f(Q_1, \dots, Q_n)) \rightarrow P \wedge f(P \wedge Q_1, \dots, P \wedge Q_n)$. We use idempotence to simplify the RHS and derive (3).

Let us introduce the conjunctive context of a term and its use in rewrite rules, informally for now. Consider a term T and the conjunction $\mathcal{C} \wedge T$ modulo idempotence of \wedge that would result from exhaustive application of rule (3) to the superterm of T . By the *conjunctive context* of T we mean the conjunction \mathcal{C} .

Example 4. The conjunctive context of the boxed occurrence of x in the term

$$(x = 3) \wedge (x^2 > y \vee (\boxed{x} = 4) \wedge U \vee V) \wedge W,$$

is $(x = 3) \wedge U \wedge W$. \square

We allow a rewrite rule $P \rightarrow T$ to refer to the conjunctive context \mathcal{C} of the rule head P . We use the following notation:

$$\mathcal{C} \setminus P \iff T.$$

This facility provides \wedge -distributivity without the undesirable effects of rule (3) on the term size.

Example 5. We can express that an equality can be used anywhere “in its scope” by viewing the equality as a conjunctive context:

$$x = a \setminus x \iff a.$$

Using this rule on the term of Example 4 results in

$$(x = 3) \wedge (3^2 > y \vee (3 = 4) \wedge U \vee V) \wedge W$$

without dissolving the disjunction. \square

1.2 Motivation and Applications

Constraint Model Simplification. Our concrete motivation behind associative commutative distributive term rewriting (ACDTR) is *constraint model mapping* as part of the G12 project [7]. A key aim of G12 is the mapping of solver independent models to efficient solver dependent models. We see ACDTR as the basis for writing these mappings. Since models are not flat conjunctions of constraints we need to go beyond AC term rewriting or CHRs.

Example 6. Consider the following simple constraint model inspired by the Social Golfers problem. For two groups g_1 and g_2 playing in the same week there can be no overlap in players: $\text{maxOverlap}(g_1, g_2, 0)$ The aim is to maximise the number of times the overlap between two groups is less than 2; in other words minimise the number of times two players play together in a group.

$$\begin{array}{ll} \text{constraint} & \bigwedge_{\substack{\forall w \in \text{Weeks} \\ \forall g_1, g_2 \in \text{weeks}[w] \\ g_1 < g_2}} \text{maxOverlap}(g_1, g_2, 0) \\ \\ \text{maximise} & \sum_{\substack{\forall w_1, w_2 \in \text{Weeks} \\ \forall g_1 \in \text{weeks}[w_1] \\ \forall g_2 \in \text{weeks}[w_2] \\ g_1 < g_2}} \text{holds}(\text{maxOverlap}(g_1, g_2, 1)) \end{array}$$

Consider the following ACDTR program for optimising this constraint model.

$$\begin{aligned} \text{maxOverlap}(a, b, c_1) \setminus \text{maxOverlap}(a, b, c_2) &\iff c_2 \geq c_1 \mid \text{true} \\ \text{holds}(\text{true}) &\iff 1 \\ \text{holds}(\text{false}) &\iff 0 \end{aligned}$$

The first rule removes redundant *maxOverlap* constraints. The next two rules implement partial evaluation of the *holds* auxiliary function which coerces a Boolean to an integer.

By representing the constraint model as a giant term, we can optimise the model by applying the ACDTR program. For example, consider the trivial case with one week and two groups G_1 and G_2 . The model becomes

$$\text{maxOverlap}(G_1, G_2, 0) \wedge \text{maximise}(\text{holds}(\text{maxOverlap}(G_1, G_2, 1))).$$

The subterm $\text{holds}(\text{maxOverlap}(G_1, G_2, 1))$ simplifies to 1 using the conjunctive context $\text{maxOverlap}(G_1, G_2, 0)$. \square

It is clear that pure CHRs are insufficient for constraint model mapping for at least two reasons, namely

- a constraint model, e.g. Example 6, is typically not a flattened conjunction;
- some rules rewrite functions, e.g. rules (2) and (3) rewriting function *holds*, which is outside the scope of CHRs (which rewrite constraints only).

Global Definitions. As we have seen conjunctive context matching provides a natural mechanism for making global information available. In a constraint model, structured data and constraint definitions are typically global, i.e. on the top level, while access to the data and the use of a defined constraint is local, e.g. the type information from Example 1. Another example is partial evaluation.

Example 7. The solver independent modelling language has support for arrays. Take a model having an array a of given values. It could be represented as the top-level term $\text{array}(a, [3, 1, 4, 1, 5, 9, 2, 7])$. Deeper inside the model, accesses to the array a occur, such as in the constraint $x > y + \text{lookup}(a, 3)$. The following rules expand such an array lookup:

$$\begin{aligned} \text{array}(A, \text{Array}) \setminus \text{lookup}(A, \text{Index}) &\iff \text{list_element}(\text{Array}, \text{Index}) \\ \text{list_element}([X|Xs], 0) &\iff X \\ \text{list_element}([X|Xs], N) &\iff N > 0 \mid \text{list_element}(Xs, N - 1) \end{aligned}$$

Referring to the respective array of the lookup expression via its conjunctive context allows us to ignore the direct context of the lookup, i.e. the concrete constraint or expression in which it occurs. \square

Propagation rules. When processing a logical formula, it is often useful to be able to specify that a new formula Q can be derived from an existing formula P *without consuming* P . In basic term rewriting, the obvious rule $P \iff P \wedge Q$ causes trivial non-termination. This issue is recognised in CHRs, which provide support for inference or *propagation* rules. We account for this fact and use rules of the form $P \implies Q$ to express such circumstances.

Example 8. The following is the classic CHR `leq` program reimplemented for ACD term rewriting (we omit the basic rules for logical connectives):

$$\begin{array}{ll} \text{leq}(X, X) \iff \text{true} & (\text{reflexivity}) \\ \text{leq}(X, Y) \setminus \text{leq}(Y, X) \iff X = Y & (\text{antisymmetry}) \\ \text{leq}(X, Y) \setminus \text{leq}(X, Y) \iff \text{true} & (\text{idempotence}) \\ \text{leq}(X, Y) \wedge \text{leq}(Y, Z) \implies \text{leq}(X, Z) & (\text{transitivity}) \end{array}$$

These rules are almost the same as the CHR version, with the exception of the second and third rule (*antisymmetry* and *idempotence*) which generalise its original by using conjunctive context matching. \square

Propagation rules are also used for adding redundant information during model mapping.

The rest of the paper is organised as follows. Section 2 covers the standard syntax and notation of term rewriting. Section 3 defines the declarative and operational semantics of ACDTR. Section 4 describes a prototype implementation of ACDTR as part of the G12 project. Section 5 compares ACDTR with related languages. Finally, in Section 6 we conclude.

2 Preliminaries

In this section we briefly introduce the notation and terminology used in this paper. Much of this is borrowed from term rewriting [3].

We use $\mathcal{T}(\Sigma, X)$ to represent the set of all terms constructed from a set of function symbols Σ and set of variables X (assumed to be countably infinite). We use $\Sigma^{(n)} \subseteq \Sigma$ to represent the set of function symbols of arity n .

A *position* is a string (sequence) of integers that uniquely determines a subterm of a term T , where ϵ represents the empty string. We define function $T|_p$, which returns the subterm of T at position p as

$$\begin{array}{l} T|_\epsilon = T \\ f(T_1, \dots, T_i, \dots, T_n)|_{ip} = T_i|_p \end{array}$$

We similarly define a function $T[S]_p$ which replaces the subterm of T at position p with term S . We define the set $\mathcal{Pos}(T)$ to represent the set of all *positions* of subterms in T .

An *identity* is a pair $(s, t) \in \mathcal{T}(\Sigma, X) \times \mathcal{T}(\Sigma, X)$, which is usually written as $s \approx t$. Given a set of identities E , we define \approx_E to be the set of identities closed under the axioms of *equational logic* [3], i.e. symmetry, transitivity, etc.

We define the congruence class $[T]_{\approx_E} = \{S \in \mathcal{T}(\Sigma, X) \mid S \approx_E T\}$ as the set of terms equal to T with respect to E .

Finally, we define function $vars(T)$ to return the set of variables in T .

3 Syntax and Semantics

The syntax of ACDTR closely resembles that of CHRs. There are three types of rules of the following form:

$$\begin{array}{ll} \text{(simplification)} & r @ H \iff g \mid B \\ \text{(propagation)} & r @ H \implies g \mid B \\ \text{(simpagation)} & r @ C \setminus H \iff g \mid B \end{array}$$

where r is a *rule identifier*, and *head* H , *conjunctive context* C , *guard* g and *body* B are arbitrary terms. The rule identifier is assumed to uniquely determine the rule. A program P is a set of rules.

We assume that $vars(g) \subseteq vars(H)$ or $vars(g) \subseteq vars(H) \cup vars(C)$ (for simpagation rules). The rule identifier can be omitted. If $g = \text{true}$ then the guard can be omitted.

We present the declarative semantics of ACDTR based on equational logic.

First we define the set of operators that ACDTR treats specially.

Definition 1 (Operators). We define the set of associate commutative operators as AC . The set AC must satisfy $AC \subseteq \Sigma^{(2)}$ and $(\wedge) \in AC$.

For our examples we assume that $AC = \{\wedge, \vee, +, \times\}$. We also treat the operator \wedge as *distributive* as explained below.

ACDTR supports a simple form of guards.

Definition 2 (Guards). A guard is a term. We denote the set of all “true” guards as \mathcal{G} , i.e. a guard g is said to hold iff $g \in \mathcal{G}$. We assume that $\text{true} \in \mathcal{G}$ and $\text{false} \notin \mathcal{G}$.

We can now define the declarative semantics for ACDTR. In order to do so we employ a special binary operator *where* to explicitly attach a conjunctive context to a term. Intuitively, the meaning of T *where* C is equivalent to that of T provided C is *true*, otherwise the meaning of T *where* C is unconstrained. For Boolean expressions, it is useful to interpret *where* as conjunction \wedge , therefore *where*-distribution, i.e. identity (6) below, becomes equivalent to \wedge -distribution (3). The advantage of distinguishing *where* and \wedge is that we are not forced to extend the definition of \wedge to arbitrary (non-Boolean) functions.

We denote by \mathcal{B} the following set of *built-in* identities:

$$A \circ B \approx B \circ A \tag{1}$$

$$(A \circ B) \circ C \approx A \circ (B \circ C) \tag{2}$$

$$T \approx (T \text{ where } \text{true}) \tag{3}$$

$$A \wedge B \approx (A \text{ where } B) \wedge B \tag{4}$$

$$T \text{ where } (W_1 \wedge W_2) \approx (T \text{ where } W_1) \text{ where } W_2 \tag{5}$$

$$f(A_1, \dots, A_i, \dots, A_n) \text{ where } W \approx f(A_1, \dots, A_i \text{ where } W, \dots, A_n) \text{ where } W \tag{6}$$

for all $\circ \in AC$, functions $f \in \Sigma^{(n)}$, and $i \in \{1, \dots, n\}$.

Definition 3 (Declarative Semantics for ACDTR). *The declarative semantics for an ACDTR program P (represented as a multiset of rules) is given by the function $\llbracket \cdot \rrbracket$ defined as follows:*

$$\begin{aligned} \llbracket P \rrbracket &= \{ \llbracket \theta(R) \rrbracket \mid \forall R, \theta . R \in P \wedge \theta(\text{guard}(R)) \in \mathcal{G} \} \cup \mathcal{B} \\ \llbracket H \iff g \mid B \rrbracket &= \exists_{\text{vars}(B) - \text{vars}(H)} (H \approx B) \\ \llbracket C \setminus H \iff g \mid B \rrbracket &= \exists_{\text{vars}(B) - \text{vars}(C, H)} (H \text{ where } C \approx B \text{ where } C) \\ \llbracket H \implies g \mid B \rrbracket &= \exists_{\text{vars}(B) - \text{vars}(H)} (H \approx H \wedge B) \end{aligned}$$

where function $\text{guard}(R)$ returns the guard of a rule.

The function $\llbracket \cdot \rrbracket$ maps ACDTR rules to identities between the head and the body terms, where body-only variables are existentially quantified.³ Note that there is a new identity for each possible binding of $\text{guard}(R)$ that holds in \mathcal{G} . A propagation rule is equivalent to a simplification rule that (re)introduces the head H (in conjunction with the body B) in the RHS. This is analogous to propagation rules under CHR.

A simpagation rule is equivalent to a simplification rule provided the conjunctive context is satisfied.

The built-in rules \mathcal{B} from Definition 3 contain identities for creating/destroying (3) and (4), combining/splitting (5), and distributing downwards/upwards (6) a conjunctive context in terms of the *where* operator.

The set \mathcal{B} also contains identities (1) and (2) for the associative/commutative properties of the AC operators.

Example 9. Consider the following ACDTR rule and the corresponding identity.

$$\llbracket X = Y \setminus X \iff Y \rrbracket = (Y \text{ where } X = Y) \approx (X \text{ where } X = Y) \quad (7)$$

Under this identity and using the rules in \mathcal{B} , we can show that $f(A) \wedge (A = B) \approx f(B) \wedge (A = B)$, as follows.

$$\begin{aligned} f(A) \wedge (A = B) &\approx_{(4)} \\ (f(A) \text{ where } (A = B)) \wedge (A = B) &\approx_{(6)} \\ (f(A \text{ where } (A = B)) \text{ where } (A = B)) \wedge (A = B) &\approx_{(7)} \\ (f(B \text{ where } (A = B)) \text{ where } (A = B)) \wedge (A = B) &\approx_{(6)} \\ (f(B) \text{ where } (A = B)) \wedge (A = B) &\approx_{(4)} \\ f(B) \wedge (A = B) & \end{aligned}$$

□

3.1 Operational Semantics

In this section we describe the operational semantics of ACDTR. It is based on the theoretical operational semantics of CHRs [1,4]. This includes support for identifiers and propagation histories, and conjunctive context matching for simpagation rules.

³ All other variables are implicitly universally quantified, where the universal quantifiers appear outside the existential ones.

Propagation history. The CHR concept of a *propagation history*, which prevents trivial non-termination of propagation rules, needs to be generalised over arbitrary terms for ACDTR. A propagation history is essentially a record of all propagation rule applications, which is checked to ensure a propagation rule is not applied twice to the same (sub)term.

In CHRs, each constraint is associated with a unique *identifier*. If multiple copies of the same constraint appear in the CHR store, then each copy is assigned a different identifier. We extend the notion of identifiers to arbitrary terms.

Definition 4 (Identifiers). *An identifier is an integer associated with each (sub)term. We use the notation $T\#i$ to indicate that term T has been associated with identifier i . A term T is annotated if T and all subterms of T are associated with an identifier. We also define function $\text{ids}(T)$ to return the set of identifiers in T , and $\text{term}(T)$ to return the non-annotated version of T .*

For example, $T = f(a\#1, b\#2)\#3$ is an annotated term, where $\text{ids}(T) = \{1, 2, 3\}$ and $\text{term}(T) = f(a, b)$.

Identifiers are considered separate from the term. We could be more precise by separating the two, i.e. explicitly maintain a map between $\text{Pos}(T)$ and the identifiers for T . We do not use this approach for space reasons. We extend and overload all of the standard operations over terms (e.g. from Section 2) to annotated terms in the obvious manner. For example, the subterm relation $T|_p$ over annotated terms returns the annotated term at position p . The exception are elements of the congruence class $[T]_{\approx_{AC}}$, formed by the AC relation \approx_{AC} , which we assume satisfies the following constraints.

$$\begin{aligned} A\#i \circ B\#j &\approx_{AC} B\#j \circ A\#i \\ A\#i \circ (B\#j \circ C\#k) &\approx_{AC} (A\#i \circ B\#j) \circ C\#k \end{aligned}$$

We have neglected to mention the identifiers over AC operators. These identifiers will be ignored later, so we leave them unconstrained.

A propagation history is a set of entries defined as follows.

Definition 5 (Entries). *A propagation history entry is of the form $(r @ E)$, where r is a propagation rule identifier, and E is a string of identifiers. We define function $\text{entry}(r, T)$ to return the propagation history entry of rule r for annotated term T as follows.*

$$\begin{aligned} \text{entry}(r, T) &= (r @ \text{entry}(T)) \\ \text{entry}(T_1 \circ T_2) &= \text{entry}(T_1) \text{entry}(T_2) && \circ \in AC \\ \text{entry}(f(T_1, \dots, T_n)\#i) &= i \text{entry}(T_1) \dots \text{entry}(T_n) && \text{otherwise} \end{aligned}$$

This definition means that propagation history entries are unaffected by associativity, but are effected by commutativity.

Example 10. Consider the annotated term $T = f((a\#1 \wedge b\#2)\#3)\#4$. We have that $T \in [T]_{\approx_{AC}}$ and $T' = f((b\#2 \wedge a\#1)\#3)\#4 \in [T]_{\approx_{AC}}$. Although T and T' belong to $[T]_{\approx_{AC}}$ they have different propagation history entries, e.g. $\text{entry}(r, T) = (r @ (4 \ 1 \ 2))$ while $\text{entry}(r, T') = (r @ (4 \ 2 \ 1))$. \square

When a (sub)term is rewritten into another, the new term is assigned a set of new unique identifiers. We define the auxiliary function $\text{annotate}(\mathcal{P}, T) = T_a$ to map a set of identifiers \mathcal{P} and un-annotated term T to an annotated term T_a such that $\text{ids}(T_a) \cap \mathcal{P} = \emptyset$ and $|\text{ids}(T_a)| = |\text{Pos}(T)|$. These conditions ensure that all identifiers are new and unique.

When a rule is applied the propagation history must be updated accordingly to reflect which terms are copied from the matching. For example, the rule $f(X) \iff g(X, X)$ essentially clones the term matching X . The identifiers, however, are not cloned. If a term is cloned, we expect that both copies will inherit the propagation history of the original. Likewise, terms can be merged, e.g. $g(X, X) \iff f(X)$ merges two instances of the term matching X . In this case, the propagation histories of the copies are also merged.

To achieve this we duplicate entries in the propagation history for each occurrence of a variable in the body that also appeared in the head.

Definition 6 (Updating History). Define function

$$\text{update}(H, H_a, B, B_a, T_0) = T_1$$

where H and B are un-annotated terms, H_a and B_a are annotated terms, and T_0 and T_1 are propagation histories. T_1 is a minimal propagation history satisfying the following conditions:

- $T_0 \subseteq T_1$;
- $\forall p \in \text{Pos}(H)$ such that $H|_p = V \in X$ (where X is the set of variables), and $\exists q \in \text{Pos}(B)$ such that $B|_q = V$, then define identifier renaming ρ such that $\rho(H_a|_p)$ and $B_a|_q$ are identical annotated terms. Then if $E \in T_0$ we have that $\rho(E) \in T_1$.

Example 11. Consider rewriting the term $H_a = f((a\#1 \wedge b\#2)\#3)\#4$ with a propagation history of $T_0 = \{(r \ @ \ (1 \ 2))\}$ using the rule $f(X) \iff g(X, X)$. The resulting term is $B_a = g((a\#5 \wedge b\#6)\#7), (a\#8 \wedge b\#9)\#10\#11$ and the new propagation history is $T_1 = \{(r \ @ \ (1 \ 2)), (r \ @ \ (5 \ 6)), (r \ @ \ (8 \ 9))\}$. \square

Conjunctive context. According to the declarative semantics, a term T with conjunctive context C is represented as $(T \text{ where } C)$. Operationally, we will never explicitly build a term containing a *where* clause. Instead we use the following function to compute the conjunctive context of a subterm on demand.

Definition 7 (Conjunctive Context). Given an (annotated) term T and a position $p \in \text{Pos}(T)$, we define function $\text{cc}(T, p)$ to return the conjunctive context at position p as follows.

$$\begin{aligned} \text{cc}(T, \epsilon) &= \text{true} \\ \text{cc}(A \wedge B, 1p) &= B \wedge \text{cc}(A, p) \\ \text{cc}(A \wedge B, 2p) &= A \wedge \text{cc}(B, p) \\ \text{cc}(f(T_1, \dots, T_i, \dots, T_n), ip) &= \text{cc}(T_i, p) \quad (f \neq \wedge) \end{aligned}$$

States and transitions. The operational semantics are defined as a set of transitions on execution states.

Definition 8 (Execution States). An execution state is a tuple of the form $\langle G, T, \mathcal{V}, \mathcal{P} \rangle$, where G is a term (the goal), T is the propagation history, \mathcal{V} is the set of variables appearing in the initial goal and \mathcal{P} is a set of identifiers.

We also define initial and final states as follows.

Definition 9 (Initial and Final States). Given an initial goal G for program P , the initial state of G is

$$\langle G_a, \emptyset, \text{vars}(G), \text{ids}(G_a) \rangle$$

where $G_a = \text{annotate}(\emptyset, G)$. A final state is a state where no more rules are applicable to the goal G .

We can now define the operational semantics of ACDTR as follows.

Definition 10 (Operational Semantics).

$$\langle G_0, T_0, \mathcal{V}, \mathcal{P}_0 \rangle \mapsto \langle G_1, T_1, \mathcal{V}, \mathcal{P}_1 \rangle$$

1. Simplify: There exists a (renamed) rule from P

$$H \iff g \mid B$$

such that there exists a matching substitution θ and a term G'_0 such that

- $G_0 \approx_{AC} G'_0$
- $\exists p \in \mathcal{Pos}(G'_0) . G'_0|_p = \theta(H)$
- $\theta(g) \in \mathcal{G}$
- $B_a = \text{annotate}(\mathcal{P}_0, \theta(B))$

Then $G_1 = G'_0[B_a]_p$, $\mathcal{P}_1 = \mathcal{P}_0 \cup \text{ids}(G_1)$ and $T_1 = \text{update}(H, G'_0|_p, B, B_a, T_0)$.

2. Propagate: There exists a (renamed) rule from P

$$r @ H \implies g \mid B$$

such that there exists a matching substitution θ and a term G'_0 such that

- $G_0 \approx_{AC} G'_0$
- $\exists p \in \mathcal{Pos}(G'_0) . G'_0|_p = \theta(H)$
- $\theta(g) \in \mathcal{G}$
- $\text{entry}(r, G'_0|_p) \notin T_0$
- $B_a = \text{annotate}(\mathcal{P}_0, \theta(B))$

Then $G_1 = G'_0[G'_0|_p \wedge B_a]_p$, $T_1 = \text{update}(H, G'_0|_p, B, B_a, T_0) \cup \{\text{entry}(r, G'_0|_p)\}$ and $\mathcal{P}_1 = \mathcal{P}_0 \cup \text{ids}(G_1)$.

3. Simpagate: There exists a (renamed) rule from P

$$C \setminus H \iff g \mid B$$

such that there exists a matching substitution θ and a term G'_0 such that

$$\begin{aligned}
& \langle (leq(X_1, Y_2)_3 \wedge leq(Y_5, Z_6)_7 \wedge \neg leq(X_{10}, Z_{11})_{12}), \emptyset \rangle \mapsto_{trans} \\
& \langle (leq(X_1, Y_2)_3 \wedge leq(Y_5, Z_6)_7 \wedge leq(X_{15}, Z_{16})_{14} \wedge \neg leq(X_{10}, Z_{11})_{12}), T \rangle \mapsto_{idemp} \\
& \langle (leq(X_1, Y_2)_3 \wedge leq(Y_5, Z_6)_7 \wedge leq(X_{15}, Z_{16})_{14} \wedge \neg true_{17}), T \rangle \mapsto_{simplify} \\
& \langle (leq(X_1, Y_2)_3 \wedge leq(Y_5, Z_6)_7 \wedge leq(X_{15}, Z_{16})_{14} \wedge false_{18}), T \rangle \mapsto_{simplify} \\
& \langle (leq(X_1, Y_2)_3 \wedge leq(Y_5, Z_6)_7 \wedge false_{19}), T \rangle \mapsto_{simplify} \\
& \langle (leq(X_1, Y_2)_3 \wedge false_{20}), T \rangle \mapsto_{simplify} \\
& \langle (false_{21}), T \rangle
\end{aligned}$$

Fig. 1. Example derivation for the *leq* program.

- $G_0 \approx_{AC} G'_0$
- $\exists p \in \mathcal{Pos}(G'_0) . G'_0|_p = \theta(H)$
- $\exists D. \theta(C) \wedge D \approx_{AC} cc(G'_0, p)$
- $\theta(g) \in \mathcal{G}$
- $B_a = \text{annotate}(\mathcal{P}_0, \theta(B))$

Then $G_1 = G'_0[B_a]_p$, $T_1 = \text{update}(H, G'_0|_p, B, B_a, T_0)$ and $\mathcal{P}_1 = \mathcal{P}_0 \cup \text{ids}(G_1)$.

Example. Consider the *leq* program from Example 8 with the goal

$$leq(X, Y) \wedge leq(Y, Z) \wedge \neg leq(X, Z)$$

Figure 1 shows one possible derivation of this goal to the final state representing *false*. For brevity, we omit the \mathcal{V} and \mathcal{P} fields, and represent identifiers as subscripts, i.e. $T\#i = T_i$. Also we substitute $T = \{\text{transitivity} @ (3\ 2\ 1\ 7\ 5\ 6)\}$.

We can state a soundness result for ACDTR.

Theorem 1 (Soundness). *If $\langle G_0, T_0, \mathcal{V}, \mathcal{P} \rangle \mapsto^* \langle G', T', \mathcal{V}, \mathcal{P}' \rangle$ with respect to a program P , then $\llbracket P \rrbracket \models \exists_{vars(G')-\mathcal{V}} G_0 \approx G'$*

This means that for all algebras \mathcal{A} that satisfy $\llbracket P \rrbracket$, G_0 and G' are equivalent for some assignment of the fresh variables in G' .

4 Implementation

We have implemented a prototype version of ACDTR as part of the mapping language of the G12 project, called Cadmium. In this section we give an overview of the implementation details. In particular, we will focus on the implementation of conjunctive context matching, which is the main contribution of this paper.

Cadmium constructs *normalised* terms from the bottom up. Here, a *normalised* term is one that cannot be reduced further by an application of a rule. Given a goal $f(t_1, \dots, t_n)$, we first must recursively normalise all of t_1, \dots, t_n (to say s_1, \dots, s_n), and then attempt to find a rule that can be applied to the top-level of $f(s_1, \dots, s_n)$. This is the standard execution algorithm used by many TRS implementations.

This approach of normalising terms bottom up is complicated by the consideration of conjunctive context matching. This is because the conjunctive context of the current term appears “higher up” in the overall goal term. Thus conjunctive context must be passed top down, yet we are normalising bottom up. This means there is no guarantee that the conjunctive context is normalised.

Example 12. Consider the following ACDTR program that uses conjunctive context matching.

$$\begin{aligned} X = V \setminus X &\iff var(X) \wedge nonvar(V) \mid V. \\ one(X) &\iff X = 1. \\ not_one(1) &\iff false. \end{aligned}$$

Consider the goal $not_one(A) \wedge one(A)$, which we expect should be normalised to *false*. Assume that the sub-term $not_one(A)$ is selected for normalisation first. The conjunctive context for $not_one(A)$ (and its subterm A) is $one(A)$. No rule is applicable, so $not_one(A)$ is not reduced.

Next the subterm $one(A)$ is reduced. The second rule will fire resulting in the new term $A = 1$. Now the conjunctive context for the first term $not_one(A)$ has changed to $A = 1$, so we expect that A should be rewritten to the number 1. However $not_one(A)$ has already been considered for normalisation. \square

The current Cadmium prototype solves this problem by re-normalising terms when and if the conjunctive context “changes”. For example, when the conjunctive context $one(A)$ changes to $A = 1$, the term $not_one(X)$ will be renormalised to $not_one(1)$ by the first rule.

The general execution algorithm for Cadmium is shown in Figure 2. Function **normalise** takes a term T , a substitution θ , a conjunctive context CC and a Boolean value Ch which keeps track of when the conjunctive context of the current subterm has changed. If $Ch = true$, then we can assume the substitution θ maps variables to normalised terms. For the initial goal, we assume θ is empty, otherwise if we are executing a body of a rule, then θ is the matching substitution.

Operationally, **normalise** splits into three cases depending on what T is. If T is a variable, and the conjunctive context has changed (i.e. $Ch = true$), then $\theta(T)$ is no longer guaranteed to be normalised. In this case we return the result of renormalising $\theta(T)$ with respect to CC . Otherwise if $Ch = false$, we simply return $\theta(T)$ which must be already normalised. If T is a conjunction $T_1 \wedge T_2$, we repeatedly call **normalise** on each conjunct with the other added to the conjunctive context. This is repeated until a fixed point (i.e. further normalisation does not result in either conjunct changing) is reached, and then return the result of **apply_rule** on the which we will discuss below. This fixed point calculation accounts for the case where the conjunctive context of a term changes, as shown in Example 12. Otherwise, if T is any other term of the form $f(T_1, \dots, T_n)$, construct the new term T' by normalising each argument. Finally we return the result of **apply_rule** applied to T' .

The function call **apply_rule**(T', CC) will attempt to apply a rule to normalised term T' with respect to conjunctive context CC . If a matching rule is found, then

```

normalise( $T, \theta, CC, Ch$ )
  if  $is\_var(T)$ 
    if  $Ch$ 
      return normalise( $\theta(T), \theta, CC, false$ )
    else
      return  $\theta(T)$ 
  else if  $T = T_1 \wedge T_2$ 
    do
       $T'_1 := T_1$ 
       $T'_2 := T_2$ 
       $T_1 := \text{normalise}(T'_1, \theta, T'_2 \wedge CC, true)$ 
       $T_2 := \text{normalise}(T'_2, \theta, T'_1 \wedge CC, true)$ 
      while  $T_1 \neq T'_1 \wedge T_2 \neq T'_2$ 
        return apply_rule( $T'_1 \wedge T'_2, CC$ )
    else
       $T = f(T_1, \dots, T_n)$ 
       $T' := f(\text{normalise}(T_1, \theta, CC, Ch), \dots, \text{normalise}(T_n, \theta, CC, Ch))$ 
      return apply_rule( $T', CC$ )

```

Fig. 2. Pseudo code of the Cadmium execution algorithm.

the result of $\text{normalise}(B, \theta, CC, false)$ is returned, where B is the (renamed) rule body and θ is the matching substitution. Otherwise, T' is simply returned.

5 Related Work

ACDTR is closely related to both TRS and CHRs, and in this section we compare the three languages.

5.1 AC Term Rewriting Systems

The problem of dealing with associative commutative operators in TRS is well studied. A popular solution is to perform the rewriting modulo some permutation of the AC operators. Although this complicates the matching algorithm, the problem of trivial non-termination (e.g. by continually rewriting with respect to commutativity) is solved.

ACDTR subsumes ACTRS (Associative Commutative TRS) in that we have introduced distributivity (via simpagation rules), and added some “CHR-style” concepts such as identifiers and propagation rules.

Given an ACTRS program, we can map it to an equivalent ACDTR program by interpreting each ACTRS rule $H \rightarrow B$ as the ACDTR rule $H \iff B$. We can now state the theorem relating ACTRS and ACDTR.

Theorem 2. *Let P be an ACTRS program and T a ground term, then $T \rightarrow^* S$ under P iff $\langle T_a, \emptyset, \emptyset, \text{ids}(T_a) \rangle \mapsto^* \langle S_a, \emptyset, \emptyset, \mathcal{P} \rangle$ under $\alpha(P)$ (where $T_a = \text{annotate}(\emptyset, T)$) for some \mathcal{P} and $\text{term}(S_a) = S$.*

5.2 CHRs and CHR^\vee

ACDTR has been deliberately designed to be an extension of CHRs. Several CHR concepts, e.g. propagation rules, etc., have been adapted.

There are differences between CHRs and ACDTR. The main difference is that ACDTR does not have a “built-in” or “underlying” solver, i.e. ACDTR is not a constraint programming language. However it is possible to encode solvers directly as rules, e.g. the simple *leq* solver from Example 8. Another important difference is that CHRs is based on predicate logic, where there exists a distinction between predicate symbols (i.e. the names of the constraints) and functions (used to construct terms). ACDTR is based on equational logic between terms, hence there is no distinction between predicates and functions (a predicate is just a Boolean function). To overcome this, we assume the existence of a set Pred , which contains the set of function symbols that are Boolean functions. We assume that $AC \cap \text{Pred} = \{\wedge^{(2)}\}$.

The mapping between a CHR program and an ACDTR program is simply $\alpha(P) = P \cup \{X \wedge \text{true} \iff X\}$.⁴ However, we assume program P is restricted as follows:

- rules have no guards apart from implicit equality guards; and
- the only built-in constraint is *true*

and the initial goal G is also restricted:

- G must be of the form $G_0 \wedge \dots \wedge G_n$ for $n > 0$;
- Each G_i is of the form $f_i(A_0, \dots, A_m)$ for $m \geq 0$ and $f_i \in \text{Pred}$;
- For all $p \in \text{Pos}(A_j)$, $0 \leq j \leq m$ we have that if $A_j|_p = g(B_0, \dots, B_q)$ then $g^{(q)} \notin AC$ and $g^{(q)} \notin \text{Pred}$.

These conditions disallow predicate symbols from appearing as arguments in CHR constraints.

Theorem 3. *Let P be a CHR program, and G an initial goal both satisfying the above conditions, then $\langle G, \emptyset, \text{true}, \emptyset \rangle_1^\vee \mapsto \langle \emptyset, S, \text{true}, T \rangle_i^\vee$ (for some T , i and $\mathcal{V} = \text{vars}(G)$) under the theoretical operational semantics [4] for CHRs iff $\langle G_a, \emptyset, \mathcal{V}, \text{ids}(G_a) \rangle \mapsto \langle S_a, T', \mathcal{V}, \mathcal{P} \rangle$ (for some T' , \mathcal{P}) under ACDTR, where $\text{term}(S_a) = S_1 \wedge \dots \wedge S_n$ and $S = \{S_1 \# i_1, \dots, S_n \# i_n\}$ for some identifiers i_1, \dots, i_n .*

We believe that Theorem 3 could be extended to include CHR programs that extend an underlying solver, provided the rules for handling tell constraints are added to the ACDTR program. For example, we can combine rules for rational tree unification with the *leq* program from Example 8 to get a program equivalent to the traditional *leq* program under CHRs.

ACDTR generalises CHRs by allowing other operators besides conjunction inside the head or body of rules. One such extension of CHRs has been studied before, namely CHR^\vee [2] which allows disjunction in the body. Unlike ACDTR,

⁴ There is one slight difference in syntax: CHRs use ‘,’ to represent conjunction, whereas ACDTR uses ‘ \wedge ’.

which manipulates disjunction syntactically, CHR^\vee typically finds solutions using backtracking search.

One notable implementation of CHR^\vee is [6], which has an operational semantics described as an and/or (\wedge/\vee) tree rewriting system. A limited form of conjunctive context matching is used, similar to that used by ACDTR, based on the knowledge that conjunction \wedge distributes over disjunction \vee . ACDTR generalises this by distributing over all functions.

6 Future Work and Conclusions

We have presented a powerful new rule-based programming language, ACDTR, that naturally extends both AC term rewriting and CHRs. The main contribution is the ability to match a rule against the conjunctive context of a (sub)term, taking advantage of the distributive property of conjunction over all possible functions. We have shown this is a natural way of expressing some problems, and by building the distributive property into the matching algorithm, we avoid non-termination issues that arise from naively implementing distribution (e.g. as rewrite rules).

We intend that ACDTR will become the theoretical basis for the Cadmium constraint mapping language as part of the G12 project [7]. Work on ACDTR and Cadmium is ongoing, and there is a wide scope for future work, such as confluence, termination and implementation/optimisation issues.

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A Examples

A.1 Further Motivating Examples

Example 13 (Conjunctive Normal Form). One of the roles of mapping models is to convert a model written in an expressive language into a restricted language which is easy to solve. Many standard approaches to solving propositional formulae require that the formulae are in conjunctive normal form (CNF). Disjunction \vee is distributive over \wedge , which can be used to establish CNF in a direct way, using the oriented rule

$$P \vee Q \wedge R \rightarrow (P \vee Q) \wedge (P \vee R).$$

CNF conversion based on this rule can exponentially increase the size of the formula. This undesirable circumstance means that in practice CNF conversions are preferred that replace subformulae by new propositional atoms, which increases the formula size at most linearly.

Let us formulate this approach in rewrite rules. To keep this example simple, we assume that the non-CNF subformula $P \vee Q \wedge R$ occurs in a positive context (for example by a preprocessing into negation normal form). We replace $Q \wedge R$ by a new atom s defined by the logical implication $s \Rightarrow (Q \wedge R)$. In rewrite rule form, we have

$$P \vee Q \wedge R \rightarrow (P \vee s) \wedge (\neg s \vee Q) \wedge (\neg s \vee R). \quad (8)$$

Unit resolution and unit subsumption can be formalised in rewrite rules. Here are two versions, one using conjunctive context and a regular one:

with conj. context:

$$P \setminus P \iff \text{true}$$

$$P \setminus \neg P \iff \text{false}$$

regular:

$$P \wedge P \rightarrow P$$

$$P \wedge (P \vee Q) \rightarrow P$$

$$P \wedge \neg P \rightarrow \text{false}$$

$$P \wedge (\neg P \vee Q) \rightarrow P \wedge Q$$

We furthermore assume rules eliminating the logical constants `true` and `false` from conjunctions and disjunctions in the obvious way. Let us contrast the two rule sets for the formula $(a \vee b \wedge (c \vee d)) \wedge d$. The following is a terminating rewrite history:

with conj. context:

$$(a \vee b \wedge (c \vee d)) \wedge d$$

$$\rightarrow (a \vee b \wedge (c \vee \text{true})) \wedge d$$

$$\rightarrow (a \vee b \wedge \text{true}) \wedge d$$

$$\rightarrow (a \vee b) \wedge d$$

regular:

$$(a \vee b \wedge (c \vee d)) \wedge d$$

$$\rightarrow (a \vee s) \wedge (\neg s \vee b) \wedge (\neg s \vee c \vee d) \wedge d$$

$$\rightarrow (a \vee s) \wedge (\neg s \vee b) \wedge \text{true} \wedge d$$

$$\rightarrow (a \vee s) \wedge (\neg s \vee b) \wedge d$$

To obtain the simple conjunct $(a \vee b)$ using the regular rule format, a rule expressing binary resolution, i.e. from $(P \vee S) \wedge (\neg S \vee Q)$ follows $(P \vee Q)$, would be required. However, such a rule is undesirable as it would create arbitrary binary resolvents, increasing formula size. Moreover, the superfluous atom s remains in the formula. \square

Example 14 (Type remapping). One of the main model mappings we are interested in expressing is where the type of a variable is changed from a high level type easy for modelling to a low level type easy to solve. A prime example of this is mapping a set variable x ranging over finite subsets of some fixed set s to an array x' of 0/1 variables indexed by s . So for variable x we have $e \in x \Leftrightarrow x'[e] = 1$. For this example we use the more concrete modelling syntax: $t : x$ indicates variable x has type t , the types we are interested are $l..u$ an integers in the range l to u , *set of* S a set ranging over elements in S , and *array* $[I]$ *of* E an array indexed by set I of elements of type E . We use *forall* and *sum* looping constructs which iterate over sets. This is expressed in ACDTR as follows.

$$\begin{array}{ll}
\text{set of } s : x \iff \text{array}[s] \text{ of } 0..1 : x' \wedge \text{map}(x, x') & (\text{typec}) \\
\text{map}(x, x') \setminus x \iff x' & (\text{vsubs}) \\
\text{array}[s] \text{ of } 0..1 : x \setminus \text{card}(x) \iff \text{sum}(e \text{ in } s) x[e] & (\text{card}) \\
\text{array}[s] \text{ of } 0..1 : x \wedge z :: (\text{array}[s] \text{ of } 0..1 : z \wedge & \\
\text{array}[s] \text{ of } 0..1 : y \setminus x \cap y \iff \text{forall}(e \text{ in } s) z[e] = x[e] \ \&\& \ y[e]) & (\text{cap}) \\
\text{array}[s] \text{ of } 0..1 : x \wedge z :: (\text{array}[s] \text{ of } 0..1 : z \wedge & \\
\text{array}[s] \text{ of } 0..1 : y \setminus x \cup y \iff \text{forall}(e \text{ in } s) z[e] = x[e] \ || \ y[e]) & (\text{cup}) \\
\text{array}[s] \text{ of } 0..1 : x \setminus x = \emptyset \iff \text{forall}(e \text{ in } s) x[e] = 0 & (\text{emptyset}) \\
\text{card}(t :: c) \iff \text{card}(t) :: c & (\uparrow \text{card}) \\
(t_1 :: c) \cup t_2 \iff t_1 \cup t_2 :: c & (\uparrow \text{cupl}) \\
t_1 \cup (t_2 :: c) \iff t_1 \cup t_2 :: c & (\uparrow \text{cupr}) \\
(t_1 :: c) \cap t_2 \iff t_1 \cap t_2 :: c & (\uparrow \text{capl}) \\
t_1 \cap (t_2 :: c) \iff t_1 \cap t_2 :: c & (\uparrow \text{capr}) \\
(t_1 :: c) = t_2 \iff t_1 = t_2 \wedge c & (\uparrow \text{eql}) \\
t_1 = (t_2 :: c) \iff t_1 = t_2 \wedge c & (\uparrow \text{eqr}) \\
(t_1 :: c) \leq t_2 \iff t_1 \leq t_2 \wedge c & (\uparrow \text{leql}) \\
t_1 \leq (t_2 :: c) \iff t_1 \leq t_2 \wedge c & (\uparrow \text{leqr}) \\
(t :: c_1) :: c_2 \iff t :: (c_1 \wedge c_2) & (\uparrow \text{cc}) \\
\text{maxOverlap}(x, y, c) \iff \text{card}(x \cap y) \leq c & (\text{maxO})
\end{array}$$

The $::$ constructor adds some local conjunctive context to an arbitrary term (like *where*) and the last 11 rules bar 1 move this context outwards to the nearest predicate scope. The last rule defines the *maxOverlap* predicate. They are used to introduce new variables z and their type and the constraints upon them. As

an example, consider the following derivation:

$$\begin{array}{lcl}
& \text{set of } 1..n : x \wedge \text{set of } 1..n : y \wedge \text{maxOverlap}(x, y, 1) & \\
\hookrightarrow_{\text{maxO}} & \text{set of } 1..n : x \wedge \text{set of } 1..n : y \wedge \text{card}(x \cap y) \leq 1 & \\
\hookrightarrow_{\text{typec}} & \text{array}[1..n] \text{ of } 0..1 : x' \wedge \text{map}(x, x') \wedge \text{set of } 1..n : y \wedge \text{card}(x \cap y) \leq 1 & \\
\hookrightarrow_{\text{vsubs}} & \text{array}[1..n] \text{ of } 0..1 : x' \wedge \text{map}(x, x') \wedge \text{set of } 1..n : y \wedge \text{card}(x' \cap y) \leq 1 & \\
\hookrightarrow_{\text{typec}} & \text{array}[1..n] \text{ of } 0..1 : x' \wedge \text{map}(x, x') \wedge \text{array}[1..n] \text{ of } 0..1 : y' \wedge \text{map}(y, y') \wedge & \\
& \text{card}(x' \cap y) \leq 1 & \\
\hookrightarrow_{\text{vsubs}} & \text{array}[1..n] \text{ of } 0..1 : x' \wedge \text{map}(x, x') \wedge \text{array}[1..n] \text{ of } 0..1 : y' \wedge \text{map}(y, y') \wedge & \\
& \text{card}(x' \cap y') \leq 1 & \\
\hookrightarrow_{\text{cap}} & \text{array}[1..n] \text{ of } 0..1 : x' \wedge \text{map}(x, x') \wedge \text{array}[1..n] \text{ of } 0..1 : y' \wedge \text{map}(y, y') \wedge & \\
& \text{card}(z :: (\text{array}[1..n] \text{ of } 0..1 : z \wedge \text{forall}(e \text{ in } 1..n) z[e] = x'[e] \&\& y'[e]) \leq 1 & \\
\hookrightarrow_{\uparrow \text{card}} & \text{array}[1..n] \text{ of } 0..1 : x' \wedge \text{map}(x, x') \wedge \text{array}[1..n] \text{ of } 0..1 : y' \wedge \text{map}(y, y') \wedge & \\
& \text{card}(z :: (\text{array}[1..n] \text{ of } 0..1 : z \wedge \text{forall}(e \text{ in } 1..n) z[e] = x'[e] \&\& y'[e]) \leq 1 & \\
\hookrightarrow_{\uparrow \text{leql}} & \text{array}[1..n] \text{ of } 0..1 : x' \wedge \text{map}(x, x') \wedge \text{array}[1..n] \text{ of } 0..1 : y' \wedge \text{map}(y, y') \wedge & \\
& \text{card}(z) \leq 1 \wedge \text{array}[1..n] \text{ of } 0..1 : z \wedge \text{forall}(e \text{ in } 1..n) z[e] = x'[e] \&\& y'[e] &
\end{array}$$

The final goal is a flat conjunction of constraints and types. It can be similarly translated into a conjunction of pseudo-Boolean constraints that can be sent to a finite domain solver, by unrolling *forall* and replacing the arrays by sequences of n variables. \square

Example 15 (Rational Tree Unification). We can directly express the rational tree unification algorithm of Colmerauer⁵ as an ACD term rewriting system.

$$\begin{array}{ll}
f(s_1, \dots, s_n) = f(t_1, \dots, t_n) \iff s_1 = t_1 \wedge \dots \wedge s_n = t_n & (\text{split}) \\
f(s_1, \dots, s_n) = g(t_1, \dots, t_m) \iff \text{false} & (\text{fail})
\end{array}$$

The (split) rule must be defined for each constructor f/n and the (fail) rule for each pair of different constructors f/n and g/m . The remaining rules are:

$$\begin{array}{ll}
x = x \iff \text{var}(x) \mid \text{true} & (\text{id}) \\
t = x \iff \text{var}(x) \wedge \text{nonvar}(t) \mid x = t & (\text{flip}) \\
x = s \setminus x = t \iff \text{var}(x) \wedge \text{nonvar}(s) \wedge \text{size}(s) \leq \text{size}(t) \mid s = t & (\text{tsubs}) \\
x = y \setminus x \iff \text{var}(x) \wedge \text{var}(y) \wedge x \not\equiv y \mid y & (\text{vsubs})
\end{array}$$

where $\text{size}(t)$ is the size of the term t in terms of number of symbols, and \equiv is syntactic identity. Even though the goals are a single conjunction of constraints, ACD is used for succinctly expressing the (vsubs) rule which replaces one variable by another in any other position.

⁵ A. Colmerauer. Prolog and Infinite Trees. *Logic Programming, APIC Studies in Data Processing (16)*. Academic Press. 1992

The following derivation illustrates the unification process in action. The underlined part show the matching elements

$$\begin{array}{ll}
& x = y \wedge \underline{f(f(x)) = x} \wedge y = f(f(f(y))) \\
\rightarrow_{flip} & x = y \wedge \underline{x = f(f(x))} \wedge y = f(f(f(y))) \\
\rightarrow_{vsubs} & \underline{x = y} \wedge y = f(f(\underline{x})) \wedge y = f(f(f(y))) \\
\rightarrow_{vsubs} & x = y \wedge \underline{y = f(f(y))} \wedge y = f(f(f(y))) \\
\rightarrow_{tsubs} & x = y \wedge y = f(f(y)) \wedge \underline{f(f(y)) = f(f(f(y)))} \\
\rightarrow_{split} & x = y \wedge y = f(f(y)) \wedge \underline{f(y) = f(f(y))} \\
\rightarrow_{split} & x = y \wedge \underline{y = f(f(y))} \wedge y = f(y) \\
\rightarrow_{tsubs} & x = y \wedge \underline{f(y) = f(f(y))} \wedge y = f(y) \\
\rightarrow_{split} & x = y \wedge \underline{y = f(y)} \wedge y = f(y) \\
\rightarrow_{tsubs} & x = y \wedge y = f(y) \wedge \underline{f(y) = f(y)} \\
\rightarrow_{split} & x = y \wedge y = f(y) \wedge \underline{f(y) = f(y)} \\
\rightarrow_{id} & x = y \wedge y = f(y) \wedge true
\end{array}$$

□

A.2 Expanded Examples

The purpose of this section is to show some example derivations under the operational semantics of ACDTR, rather than high-level descriptions. We allow for some shorthand, namely $T \# i = T_i$.

Identifiers and conjunctive context. In this section we explain parts of the derivation from Example 15 in more detail. The initial goal is

$$x = y \wedge f(f(x)) = x \wedge y = f(f(f(y)))$$

which corresponds to the initial state:

$$\langle (((x_1 = y_2)_3 \wedge (f(f(x_4)_5)_6 = x_7)_8)_9 \wedge (y_{10} = f(f(f(y_{11})_{12})_{13})_{14})_{15})_{16}, \emptyset, \{x, y\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \rangle$$

The initial state is a quadruple contained an annotated version of the goal, an empty propagation history, the set of variables in the goal and a set of “used” identifiers.

The first derivation step is a **Simplify** transition with the *flip* rule:

$$\begin{array}{c}
\langle (((x_1 = y_2)_3 \wedge (f(f(x_4)_5)_6 = x_7)_8)_9 \wedge (y_{10} = f(f(f(y_{11})_{12})_{13})_{14})_{15})_{16}, \emptyset, \{x, y\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \rangle \\
\rightarrow \\
\langle (((x_1 = y_2)_3 \wedge (x_{17} = f(f(x_{18})_{19})_{20})_{21})_9 \wedge (y_{10} = f(f(f(y_{11})_{12})_{13})_{14})_{15})_{16}, \emptyset, \{x, y\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\} \rangle
\end{array}$$

We have replaced the annotated subterm $(f(f(x_4)_5)_6 = x_7)_8$ with $x_{17} = f(f(x_{18})_{19})_{20})_{21}$ (i.e. flipped the operands to the equality) and reannotated the

The next derivation step is a *Simpagate* transition with the *vsubs* rule.

The conjunctive context for subterm x_{17} is

$$\text{cc}(G_a, p) = (x_1 = y_2)_3 \wedge (y_{10} = f(f(f(y_{11})_{12})_{13})_{14})_{15} \wedge \text{true}$$

where G_a is the current goal and p is the position of x_{17} . The first conjunct matches the conjunctive context of the *vsubs* rule, thus subterm x_{17} is replaced with y_{21} . Identifier 21 is added to the list of used identifiers.

Execution proceeds until the final state

$$\langle (x = y \wedge y = f(y)) \wedge \text{true}, \emptyset, \{x, y\}, \mathcal{P} \rangle$$

is reached, for some annotation of the goal and some set of identifiers \mathcal{P} . This is a final state because no more rules are applicable to it.

AC matching and propagation histories. Consider the propagation rule from the *leq* program:

$$trans @ leq(X, Y) \wedge leq(Y, Z) \implies X \not\equiv Y \wedge Y \not\equiv Z \mid leq(X, Z)$$

and the initial state

$$\langle leq(A_1, B_2)_3 \wedge_4 leq(B_5, A_6)_7, \emptyset, \{A, B\}, \{1, 2, 3, 4, 5, 6, 7\} \rangle.$$

We can apply **Propagate** directly (i.e. without permuting the conjunction) to arrive at the state:

$$\langle (leq(A_1, B_2)_3 \wedge_4 leq(B_5, A_6)_7) \wedge_8 leq(A_9, A_{10})_{11}, \\ \{trans @ (3 \ 1 \ 2 \ 7 \ 6 \ 5)\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \rangle.$$

The propagation history prevents the rule from firing on the same terms again, however we can permute the terms to find a new matching. Namely, we can permute the annotated goal (which we call G_a)

$$(leq(A_1, B_2)_3 \wedge_4 leq(B_5, A_6)_7) \wedge_8 leq(A_9, A_{10})_{11}$$

to

$$(leq(B_5, A_6)_7 \wedge_4 leq(A_1, B_2)_3) \wedge_8 leq(A_9, A_{10})_{11}.$$

The latter is an element of $[G_a]_{AC}$, and the identifiers have been preserved in the correct way. The entry $trans @ (7\ 6\ 5\ 3\ 1\ 2)$ is not in the propagation history, so we can apply **Propagate** again to arrive at:

$$\langle (leq(B_5, A_6)_7 \wedge_4 leq(A_1, B_2)_3) \wedge_{12} leq(B_{13}, B_{14})_{15} \rangle \wedge_8 leq(A_9, A_{10})_{11}, \\ \{trans @ (3\ 1\ 2\ 7\ 6\ 5), trans @ (7\ 6\ 5\ 3\ 1\ 2)\}, \{1...15\}.$$

Now the propagation history prevents the rule $trans$ being applied to the first two leq constraints. The guard also prevents the $trans$ rule firing on either of the two new constraints,⁶ thus we have reached a final state.

Updating propagation histories. Consider a modified version of the previous example, now with two rules:

$$X \wedge X \iff X \\ trans @ leq(X, Y) \wedge leq(Y, Z) \implies leq(X, Z)$$

The first rule enforces *idempotence* of conjunction.

Consider the initial state:

$$\langle leq(A_1, A_2)_3 \wedge_4 leq(A_5, A_6)_7 \wedge_8 leq(A_9, A_{10})_{11} \rangle, \emptyset, \{A\}, \{1...11\}$$

We apply the $trans$ rule to the first two copies of the leq constraint (with identifiers 3 and 7).

$$\langle leq(A_1, A_2)_3 \wedge_4 leq(A_5, A_6)_7 \wedge_8 leq(A_9, A_{10})_{11} \wedge_{12} leq(A_{13}, A_{14})_{15} \rangle, \\ \{trans @ (3\ 1\ 2\ 7\ 5\ 6)\}, \{A\}, \{1...15\}$$

Next we apply idempotence to leq constraints with identifiers 7 and 11.

$$\langle leq(A_1, A_2)_3 \wedge_4 leq(A_{16}, A_{17})_{18} \wedge_{12} leq(A_{13}, A_{14})_{15} \rangle, \\ \{trans @ (3\ 1\ 2\ 7\ 5\ 6), trans @ (3\ 1\ 2\ 18\ 16\ 17)\}, \{A\}, \{1...18\}$$

An extra entry ($trans @ (3\ 1\ 2\ 18\ 16\ 17)$) is added to the propagation history in order to satisfy the requirements of Definition 6. This is because we have replaced the annotated constraint $leq(A_5, A_6)_7$ with the newly annotated term $leq(A_{16}, A_{17})_{18}$, which defines an identifier renaming

$$\rho = \{5 \mapsto 16, 6 \mapsto 17, 7 \mapsto 18\}.$$

Since $E = (trans @ (3\ 1\ 2\ 7\ 5\ 6))$ is an element of the propagation history, we have that $\rho(E) = (trans @ (3\ 1\ 2\ 18\ 16\ 17))$ must also be an element, and hence the history is expanded.

⁶ Without the guard both ACDTR and CHRs are not guaranteed to terminate.